Theorem-4 Standard form of a self-Adjoint Equation:

Consider the 2nd order linear differential Equation

$$L_2(y) = a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \quad \text{—} (1)$$

It is self Adjoint, then $a_1(x) = a_0'(x)$

putting this in (1), we get

$$L_2(y) = a_0(x)\frac{d^2y}{dx^2} + a_0'(x)\frac{dy}{dx} + a_2(x)y = 0$$

$$= \frac{d}{dx}\left(a_0(x)\frac{dy}{dx}\right) + a_2(x)y = 0$$

$$= \frac{d}{dx}\left(P(x)\frac{dy}{dx}\right) + q(x)y = 0$$

where $p(x) = a_0(x)$

$q(x) = a_2(x)$

This is called strumian equation

Note : 1. $P(x)$ must be +ve

If $p(x)$ is negative, then make it +ve by multiplying with (-1).

then $p(x)$ becomes +ve

2. Reduction of equation (1) to standard self-Adjoint Equation:

$$L_2(y) = a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = 0$$

Dividing by $a_0(x)$, we get

$$\frac{d^2y}{dx^2} + \frac{a_1(x)}{a_0(x)}\frac{dy}{dx} + \frac{a_2(x)}{a_0(x)}\,y = 0$$

which is of the type

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)\,y = 0$$

Multiplying by $e^{\int P(x)\,dx}$:

$$e^{\int p(x)\,dx} \cdot \frac{d^2y}{dx^2} + e^{\int p(x)\,dx} \cdot p(x)\frac{dy}{dx} + e^{\int p(x)\,dx} \cdot q(x)\,y = 0$$

$$\frac{d}{dx}\left(e^{\int p(x)\,dx}\frac{dy}{dx}\right) + e^{\int p(x)\,dx} \cdot q(x)\,y = 0$$

Put $e^{\int p(x)\,dx} = P(x)$

and $e^{\int p(x)\,dx} \cdot q(x) = Q(x)$

$\therefore$ we get

$$\frac{d}{dx}\left(P(x)\frac{dy}{dx}\right) + Q(x)\,y = 0 \quad (1)$$

which is required solution form of given equation (1)

Note :- Legendre's equation is self - Adjoint

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)\,y = 0$$

This is self Adjoint

$\therefore a_0'(x) = a_1(x)$

Its Standard Self - Adjoint is :

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n+1)y = 0$$

Ex :- Reduce the equation to the standard Self-Adjoint form

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + \lambda y = 0$$

→ Dividing the given eqn by $(1-x^2)$:

$$\frac{d^2y}{dx^2} - \frac{x}{(1-x^2)}\frac{dy}{dx} + \frac{\lambda}{(1-x^2)}y = 0 \qquad \text{--- (1)}$$

$$I.F = e^{\int P(x)\,dx}$$

$$= e^{\int \frac{-x}{(1-x^2)}dx}$$

$$= e^{\frac{1}{2}\int \frac{-2x}{(1-x^2)}dx}$$

$$= e^{\frac{1}{2}\log(1-x^2)}$$

$$= e^{\log(1-x^2)^{1/2}}$$

$$= \sqrt{1-x^2}$$

Multiplying ① by $\sqrt{1-x^2}$

$$\sqrt{1-x^2}\,\frac{d^2y}{dx^2} - \sqrt{1-x^2}\cdot\frac{x}{(1-x^2)}\frac{dy}{dx} + \sqrt{1-x^2}\cdot\frac{\lambda}{(1-x^2)}y = 0$$

$$\sqrt{1-x^2}\,\frac{d^2y}{dx^2} - \frac{x}{\sqrt{1-x^2}}\frac{dy}{dx} + \frac{\lambda}{\sqrt{1-x^2}}y = 0$$

$$\frac{d}{dx}\left[\sqrt{1-x^2}\,\frac{dy}{dx}\right] + \frac{\lambda}{\sqrt{1-x^2}}y = 0.$$

Which is reqd. Self Adjoint form of given equation.